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Extension of the Titchmarsh representation of the Mertens function $M(x)$, and numerical support of the Riemann hypothesis

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ABSTRACT

A closed form representation of the Mertens function, without assuming simplicity of the non-trivial zeros of the zeta function, is proved and the Titchmarsh representation of the function is recovered as a special case. We exploit a representation of the Mertens function and provide numerical support of the Riemann hypothesis and the simplicity of the zeros of the zeta function.

1. Introduction

The Mertens function is defined by [1, p. 315]

$$M(x) := \sum_{n \le x} \mu(n), \tag{1.1}$$

where $\mu(x)$ is the Möbius function [2, p. 217]. In particular, we have

$$M(n) = \sum_{k=1}^{n} \mu(k). \tag{1.2}$$

It is important to note that no analytic formula for the function $M(x)$ seems to be known. However, in 1944 Titchmarsh [1, p. 318] proved the following theorem:

Statement: There is a sequence T_ν, $\nu \le T_\nu \le \nu + 1$, such that ($\rho = \sigma + \mathrm{i}\gamma$, $0 < \sigma < 1$)

$$M(x) = -2 + \lim_{\nu \to \infty} \sum_{|\gamma| < T_\nu} \frac{x^\rho}{\rho \zeta^{(1)}(\rho)} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \left(\frac{2\pi}{x}\right)^{2n}}{(2n)! n \zeta(2n+1)} \tag{1.3}$$

if x is not an integer. If x is an integer, $M(x)$ is to be replaced by $M(x) - \frac{1}{2}\mu(x)$, provided the zeros of the zeta function $\zeta(s)$ are simple.

It is to be noted that the Titchmarsh proof assumes the simplicity of the non-trivial zeros.

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We prove a representation of the Mertens function without assuming the simplicity of the zeta zeros and deduce the result (1.3) as a corollary. Using Mathematica, the representation is exploited further to give numerical support for the validity of the Riemann hypothesis and that of the simplicity of the zeros.

2. The main result

In order to state our result, we define

$$\zeta^*(\rho) := x^{-\rho}\operatorname{Re}s\left[\frac{x^s}{s\zeta(s)};\rho\right]. \tag{2.1}$$

We note that if the zero ρ is simple, then we have

$$\zeta^*(\rho) = x^{-\rho}\operatorname{Re}s\left[\frac{x^s}{s\zeta(s)};\rho\right] = \frac{1}{\rho\zeta^{(1)}(\rho)}. \tag{2.2}$$

Theorem. *The Mertens function has the closed form representation*

$$M(x) = -2 + \sum_{\rho}\zeta^*(\rho)x^{\rho} + \sum_{n=1}^{\infty}\zeta^*(-2n)x^{-2n}, \tag{2.3}$$

where the first summation is over all non-trivial zeros of the zeta function.

Note. A proof of the above theorem will be given in the next section. The sum over the non-trivial zeros ρ of $\zeta(s)$ is to be understood in the symmetric sense [3, p. 104]. We state some of immediate consequences as follows:

Corollary. *If all non-trivial zeros of the zeta function are simple, then*

$$M(x) = -2 + \sum_{\rho}\frac{1}{\rho\zeta^{(1)}(\rho)}x^{\rho} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n\zeta^{(1)}(-2n)}x^{-2n}, \tag{2.4}$$

where the first summation runs over all non-trivial zeros of the zeta function.

Proof. Follows from (2.2) and (2.3). □

Remark. The Titchmarsh representation (1.3) follows from (2.3) on assuming the simplicity of the zeros of the zeta function and using the relation [4, p. 295]

$$\zeta^{(1)}(-2n) = (-1)^n\frac{(2n)!}{2(2\pi)^{2n}}\zeta(2n+1) \quad (n = 1, 2, 3, \ldots). \tag{2.5}$$

3. Proof of the main theorem

Let $H(x)$ be the unit step function defined at zero by $H(0) := 1$. Then function (1.1) can be rewritten to give

$$M(x) = \sum_{n=1}^{\infty}\mu(n)H\left(1-\frac{n}{x}\right) \quad (x \geq 0). \tag{3.1}$$

The Möbius inversion formula [2, p. 217] leads to

$$H\left(1-\frac{1}{x}\right) = \sum_{n=1}^{\infty}M\left(\frac{x}{n}\right) \quad (x \geq 0). \tag{3.2}$$

Taking the Mellin transform of both sides in (3.2) in the variable $-s$, we have

$$\frac{1}{s} = \zeta(s)\int_0^{\infty}x^{-s-1}M(x)\mathrm{d}x \quad (s = \sigma + \mathrm{i}\tau, \sigma > 1), \tag{3.3}$$

which leads to the inverse Mellin transform representation [2, p. 260]

$$M(x) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{x^s}{s\zeta(s)}\mathrm{d}s \quad (c \geq c_0 > 1). \tag{3.4}$$

The poles of the integrand in (3.4) lie to the LHS of the line of integration. Since the zeta function is bounded in the region $\sigma \geq \sigma_0 > 1$, the inverse Mellin transform in (3.4) can be evaluated [1, pp. 318–319] by using Cauchy's residue theorem. Taking the sum of the residues at the poles at $s = 0$, at the trivial zeros $s = -2n$ $(n = 1, 2, 3, \ldots)$ and at the non-trivial zeros $s = \rho$ of the zeta function leads to (2.3), where we use that $\zeta(0) = -1/2$. Hence the proof.

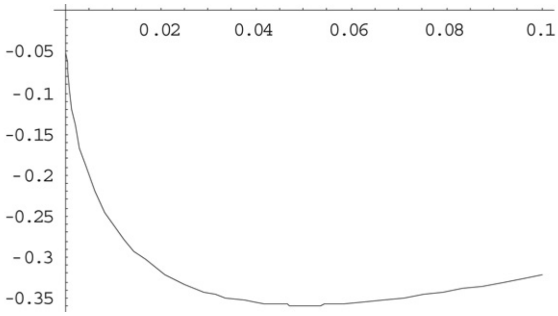

Fig. 1. The plot of the function $sF(s)$ for small values of s.

Remark. Representation (3.2) is important. A simple application of the Möbius inversion formula (see [2, p. 218]) leads to an important known result:

$$\sum_{n=1}^{\infty} M\left(\frac{x}{n}\right) = H\left(1-\frac{1}{x}\right) = 1 \quad (\forall x > 1), \tag{3.5}$$

for the series involving the Mertens function.

4. Numerical support for the validity of the Riemann hypothesis

According to [1, p. 315], a necessary and sufficient condition for the Riemann hypothesis is

$$M(x) = \mathrm{O}\left(x^{\frac{1}{2}+\varepsilon}\right) \quad (x \to \infty, \forall \varepsilon > 0). \tag{4.1}$$

However, a necessary and sufficient condition for the Riemann hypothesis and the simplicity of the zeros is that [1, p. 315]

$$M(x) = \mathrm{O}\left(x^{\frac{1}{2}}\right) \quad (x \to \infty). \tag{4.2}$$

We define a function

$$f(x) := x^{-\frac{1}{2}} M(x) \quad (x > 0). \tag{4.3}$$

Using representation (3.1), we get

$$f(x) := x^{-\frac{1}{2}} M(x) = \sum_{n=1}^{\infty} \mu(n) x^{-\frac{1}{2}} H\left(1-\frac{n}{x}\right) \quad (x > 0). \tag{4.4}$$

Note that the function $f(x)$ is discontinuous and does not seem to be suitable for asymptotic analysis. However, the Laplace transform of $f(x)$ (which is continuous and is suitable for asymptotic analysis) is found in terms of the incomplete gamma function as

$$F(s) := L[f(x); s] = \sum_{n=1}^{\pi} \mu(n) \left(s^{-\frac{1}{2}} \Gamma(1/2, ns)\right) \quad (s > 0). \tag{4.5}$$

In order to prove the Riemann hypothesis and the simplicity of the zeros of the zeta function, it would be sufficient (in view of the Final Value Theorem (FVT)) to show that

$$sF(s) = \mathrm{O}(1) \quad (s \to 0^{+}). \tag{4.6}$$

An analytic proof of (4.6) would resolve the Riemann hypothesis and the simplicity of the zeros of the zeta function. As a numerical check of (4.6), we used Mathematica to plot the above function for small values of s (Fig. 1) by taking the sum of the first ten thousand terms of the series

$$sF(s) = \sum_{n=1}^{\infty} \mu(n) \left(s^{\frac{1}{2}} \Gamma(1/2, ns)\right) \quad (s > 0). \tag{4.7}$$

The plot is in agreement with (4.6) supporting numerically the Riemann hypothesis and the simplicity of the zeros of the zeta function. It is to be remarked that for all $s \geq s_0 > 0$ [4, p. 80],

$$\sqrt{s}\Gamma(1/2, ns) = (\sqrt{\pi s})\mathrm{erfc}(\sqrt{ns}) \sim \frac{1}{\sqrt{n}} \mathrm{e}^{-ns} \quad (n \to \infty). \tag{4.8}$$

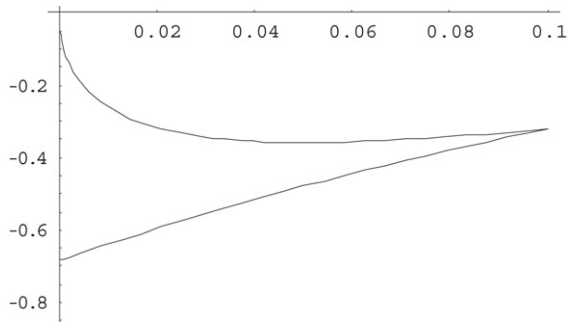

Fig. 2. The plot of the functions $tF(t)$ and $\Theta(t)$ for small values of t.

Hence, for a given $s > 0$, we can choose n sufficiently large to make the sum (4.7) and the graph (see Fig. 1) sufficiently accurate. This provides a numerical justification for the validity of the graph. We plot the functions $\Theta(t) := \sum_{n=1}^{\infty} \frac{\mu(n)}{\sqrt{n}} \mathrm{e}^{-nt}$ (obtained from (4.8)) and $tF(t)$ together in Fig. 2 and find that both the functions remain bounded for sufficiently small values of t.

5. Concluding remarks

The above approximating function $\Theta(t) := \sum_{n=1}^{\infty} \frac{\mu(n)}{\sqrt{n}} \mathrm{e}^{-nt}$ could also be used independently to support the Riemann hypothesis numerically. Firstly, Fig. 2 supports numerically the Riemann hypothesis and simplicity of the zeros of the zeta function. Secondly, we note that the function has the Mellin transform

$$\Theta_M(s) := M[\Theta(t); s] = \frac{\Gamma(s)}{\zeta(s+1/2)} \quad (\sigma > 1/2), \tag{5.1}$$

which leads to the inverse Mellin transform representation

$$\Theta(t) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(s)}{\zeta(1/2+s)} t^{-s} \mathrm{d}s \quad (c \geq c_0 > 1/2). \tag{5.2}$$

The integrand in (5.2) has three sets of poles to the LHS of the line of integration: the simple poles at $s = -n$ ($n = 0, 1, 2, 3, \ldots$) due to the gamma function, the simple poles at the trivial zeros at $s = -2n - 1/2$ ($n = 1, 2, 3, \ldots$) of $\zeta(1/2+s)$ and the poles at the non-trivial zeros at $s = \rho - 1/2$ of the above function. Let us define

$$\mathbb{R}_\rho(t) := t^{\rho-1/2} \mathrm{Re}\, s \left[\frac{\Gamma(s)}{\zeta(s+1/2)} t^{-s}; \rho - 1/2 \right]. \tag{5.3}$$

It is to be noted that if the zero ρ of the zeta function is simple, we have $\mathbb{R}_\rho = \frac{\Gamma(\rho-1/2)}{\zeta^{(1)}(\rho)}$. However, for the zero of multiplicity n_ρ, the $\mathbb{R}_\rho$ is a polynomial in $\ln t$ of degree $\leq n_\rho - 1$. Consider the above integral (5.2) around the rectangle with vertices at $c \pm \mathrm{i}T$ and $-a \pm \mathrm{i}T$ ($a > 1, c \geq c_0 > 1/2$) such that T and a have such values that no zero of the zeta function lies on the contour. Since [4, p. 6 (1.45)]

$$\Gamma(\sigma \pm \mathrm{i}\tau) = \sqrt{2\pi}\, |\tau|^{\sigma-1/2} \exp\left(-\frac{\pi}{2}|\tau|\right)(1 + \mathrm{O}(1/|\tau|)) \quad (|\tau| \to \infty), \tag{5.4}$$

the contribution from the lower and upper sides of the rectangle tends to zero as $T \to \infty$; and so we may displace the path to the left over the poles of the integrand leading to the relation (for small values of t)

$$\Theta(t) \sim \sum_{n=0}^{\infty} \frac{(-t)^n}{n!\zeta(1/2-n)} - \pi\sqrt{t} \sum_{n=1}^{\infty} \frac{(t)^{2n}}{\Gamma(2n+3/2)} + \sum_{\rho} \mathbb{R}_\rho t^{\frac{1}{2}-\rho}. \tag{5.5}$$

Therefore, for small values of t, we have asymptotically

$$\Theta(t) = \frac{1}{\zeta(1/2)} + \sum_{\rho} \mathbb{R}_\rho t^{\frac{1}{2}-\rho} + \mathrm{O}(t) \quad (t \to 0). \tag{5.6}$$

Writing

$$\Theta(t) := \sum_{n=1}^{M-1} \frac{\mu(n)}{\sqrt{n}} \mathrm{e}^{-nt} + \sum_{n=M}^{\infty} \frac{\mu(n)}{\sqrt{n}} \mathrm{e}^{-nt}, \tag{5.7}$$

we note that [4, p. 125 (3.20)] ($\sqrt{tM} > 2.7$ and $M \to \infty, t \geq t_0 > 0$)

$$\int_M^\infty \frac{1}{\sqrt{n}} e^{-nt} dn = \frac{1}{\sqrt{t}} \Gamma(1/2, tM) = \frac{\sqrt{\pi}}{\sqrt{t}} \operatorname{erfc}(\sqrt{tM}) \sim (1.132\sqrt{\pi}) \frac{\sqrt{M} e^{-tM}}{1 + 2tM} \to 0, \tag{5.8}$$

provided that $M \to \infty$, $t \geq t_0 > 0$. This supports the numerical observation that the sum $\Theta(t) := \sum_{n=1}^{\infty} \frac{\mu(n)}{\sqrt{n}} e^{-nt}$ in (5.6) remains bounded for small values of t. Thus, in view of (5.6), we have numerical support for the Riemann hypothesis being true and the zeros of the zeta function being simple.

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